LIMITS

Concept of RHL and LHL

$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$ **Left Hand Limit**

$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$ **Right Hand Limit**

if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = l$ (finite)

then, $\lim_{x \to a} f(x) = l$

Indeterminate Forms

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \times \infty, 0^0, \infty^0, 1^\infty$

Methods of Evaluating Limits

1. Factorisation Method Example

$$\lim_{x \to 2} \frac{x^3 - 2x - 4}{x^2 - 3x + 2} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 2)}{(x-2)(x-1)}$$

$$= \lim_{x \to 2} \frac{(x^2 + 2x + 2)}{(x-1)} = \frac{4 + 4 + 2}{1} = 10$$

2. Rationalisation Method Example

Used when either numerator or denominator or even both involve square roots

$$\lim_{x\to 4}\frac{\sqrt{4x+9}-5}{x-4}=\lim_{x\to 4}\frac{\sqrt{4x+9}-5}{x-4}\times\frac{\sqrt{4x+9}+5}{\sqrt{4x+9}+5}$$

$$=\lim_{x\to 4}\frac{4x-16}{(x-4)\sqrt{4x+9}+5}=\lim_{x\to 4}\frac{4(x-4)}{(x-4)\sqrt{4x+9}+5}$$

$$=\frac{4}{10}=\frac{2}{5}$$

What if x → ∞ (Limits tending to infinity)

Just take out the biggest terms common from numerator and denominator

$$\lim_{x\to\infty}\frac{2x^3-3x^2+1}{4x^3+5x^2+3}=\lim_{x\to\infty}\frac{x^3\left(2-\frac{3}{x}+\frac{1}{x^2}\right)}{x^3\left(4+\frac{5}{x}+\frac{3}{x^2}\right)}=\frac{2}{4}=\frac{1}{2}\ \left(\frac{1}{x}\to 0\right)$$

3. Cases relating when x → ∞

if a and b are leading coefficients of polynomials p(x) and q(x) with degrees m and n. Then, p(x)/q(x) = when x → ∞

a/b, if m=n	**0,** if m < n
∞, if m > n & a/b > 0	**-∞,** if m > n & a/b < 0

For evaluating the limit, when x → -∞, just put x = -t and make t, when x → ∞

Algebra Of Limits

(a) $\lim\limits_{x\to a}(f+g)(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

(b) $\lim\limits_{x\to a}(f-g)(x) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

(c) $\lim\limits_{x\to a}(c\cdot f)(x) = c\lim\limits_{x\to a} f(x)$ [c is a constant]

(d) $\lim\limits_{x\to a}(fg)(x) = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$

(e) $\lim\limits_{x\to a}\left(\frac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

(f) $\lim\limits_{x\to a}\left(f(x)\right)^n = \left(\lim\limits_{x\to a} f(x)\right)^n$

(g) $\lim\limits_{x\to a}\left(f(x)\right)^{g(x)} = \left(\lim\limits_{x\to a} f(x)\right)^{\lim\limits_{x\to a} g(x)}$

(h) If f(x)g(x) xa then $\lim\limits_{x\to a} f(x) \le \lim\limits_{x\to a} g(x)$

NOTE

- Whenever the function under consideration has one of the following traits, always check RHL & LHL for existence of limits
  - It has √, [.], {.} or mod.
  - It's piecewise defined
  - It has $a^{1/x}$ and $x\to\infty$

Some Important Results

$\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$	$\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}=\dfrac{m}{n}a^{m-n}$

Trignometric Limits

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\tan x}{x}=1=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}$$

$\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$	$(\sin x)/x$ is slightly less than 1 $(\tan x)/x$ is slightly more than 1

Using above observation,

$$\lim_{x\to 0}\left[\frac{\sin x}{x}\right]=0 \text{ whereas } \lim_{x\to 0}\left[\frac{\tan x}{x}\right]=1$$

Logarithmic and exponential Limits

$\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$	$\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\dfrac{1}{\ln a}$
$\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$	$\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$

Standard Result for 1^∞

$$\lim_{x\to a}(1+f(x))^{g(x)}=e^{\lim\limits_{x\to a}f(x).g(x)}$$

L'Hospitals Rule

- if f(x) and g(x) are differentiable functions such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ is of the format **0/0 or ∞/∞.** Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Some Important Functions

$$(1+x)^n = \left\{1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}x^3 + ...\right\}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{3!} + ...$$

$$a^x = 1 + x(\log a) + \frac{x^2}{2!}(\log a)^2 + ...$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \quad (|x|<1)$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ......$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Some Important Functions

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$$

$$a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots\ldots\ldots$$

$$\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \ldots.(-1 < x < 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots.. \ (-1 < x < 1)$$